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Abstract

We prove the existence of an optimal growth path in an economy where goods that are not consumed are either invested in next period capital or in R&D with overall non-convex production possibilities, using results of non-linear functional analysis in weighted L_p spaces of Chichilnisky (*Journal of Optimization Theory and Applications*, 1977, 61, no. 2, 504–520) and (*Journal of Mathematical Economics*, 1981, 8, 1–14).

Keywords: Endogenous growth; Technical change; Increasing returns

JEL classification: O03; O04

1. Introduction

A classical formulation explains economic growth through the accumulation of capital and from technical change. Technical change occurs through time in response to accumulated knowledge, obtained, for example, from investment in research and development (R&D). In an economy with technical change there is, therefore, a choice between investment in next period capital, and investment in a research sector to increase next period's productivity. In such an economy, each unit of capital becomes more productive with R&D, so that across time there may be increasing returns to scale. This paper defines an infinitely lived dynamic economy with these characteristics, and establishes reasonable conditions which ensure the existence of an optimal path of consumption, accumulation of capital and investment in R&D.

Mathematically, the problem is one of maximizing a non-linear functional over an infinite dimensional function space, the space of feasible paths, which is generally non-convex. The feasible paths satisfy a difference equation representing capital accumulation, R&D invest-

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ment and technological change through time. The problem is of interest because technology is endogeneously determined, and because of its possible non-convexity, see, for example, Gruenwald (1992). The standard conditions for existence, which rely on convexity (e.g. Von Weisacker, 1965), do not apply. Classical pieces on growth without convexities, such as Weitzman (1970) and Chichilnisky (1977, 1981), do not allow a choice of R&D investment that endogenously determines technological change. Furthermore, standard compactness–continuity arguments do not work here because the logarithmic utility function that we use is ill-defined on some regions of the space that include paths with some zero components, and, in addition, because the feasible set is unbounded, as it may grow through time in response to technical change. Further difficulties arise from the infinite dimensionality of the space of growth paths, because in such spaces closed bounded sets are generally not compact.

The strategy of the proof of existence is as follows. We find a growth path B that grows exponentially, and that is not feasible. This path is useful because it bounds all feasible paths in the economy. Using this bound we define a finite measure on the set of integers and consider a 'weighted' Banach space H_λ of all sequences that are summable with respect to this measure. This space includes all bounded paths, and many exponentially growing paths; in particular, it contains all the feasible growth paths in our economy. Therefore, without loss of generality, we consider the problem of maximizing a welfare function in the space H_λ, restricted to the set of all feasible growth paths. For this it would suffice to prove that the set of all feasible paths is compact, and the welfare function is continuous on it in the norm of the space H_λ. However, these standard arguments do not work here, because the logarithmic utility function is undefined over certain paths, and because the closed bounded subset of H_λ, which includes all the feasible growth paths, is not closed and therefore is not compact. We resolve this problem by finding another closed subset of the space H_λ that contains all feasible paths yielding utility values that exceed a minimal level. The main argument involves proving that this set is compact and that, when restricted to this set, the welfare function is norm-continuous. The existence theorem then shows that this set contains an optimum of our problem, thus establishing existence of an optimal growth path with endogenous technical change. The techniques utilized here rely on weighted L_p spaces, which were introduced in the theory of optimal growth in Chichilnisky (1977, 1981), and in Chichilnisky and Kalman (1980).

2. The model

The problem is to maximize the discounted logarithm of consumption over an infinite time horizon:

$$\max_{C=\{C_t\}_{t=0}^{\infty}} W(C) = \sum_{t=0}^{\infty} \beta^{-t} \ln C_t \,, \tag{1}$$

over the space of sequences $\{C_t\}_{t=0}^{\infty}$ that satisfy

$$\eta_t K_t = C_t + Z_t + K_{t+1} \,, \tag{2}$$

$$\eta_{t+1} = \eta_t g\left[\gamma \frac{Z_t}{\eta_t K_t}\right], \tag{3}$$

$$\eta(0) = \eta_0 , \qquad K(0) = K_0 ,$$

$$0 < \eta_0, K_0 < \infty, \beta > 1 , \tag{4}$$

$$\eta_0 g(\gamma) > 1 , \tag{5}$$

where

C_t = consumption in period t,
K_t = capital stock in period t (which fully depreciates),
η_t = capital productivity in period t,
Z_t = research allocation in period t (used to increase η_{t+1}),
$g(\cdot)$ = research sector function,
γ = research productivity parameter,
β = discount factor.

The following assumptions are made on the function $g(\cdot)$:

Assumption 1. $g(0) = 1$.

Assumption 2. $g'(\cdot) > 0$.

Assumption 3. g is C^1 on the interval (o, γ).

The *feasible consumption set* for an initial $\eta_0 K_0$ is therefore

$$F(\eta_0 K_0) = \{\{C_t\}_{t=0}^{\infty},\quad C_t \in R^+ \text{ s.t. there exists } \{\eta_t, \eta_{t+1}, K_t, K_{t+1}, Z_t\} \text{ satisfying (2) to (5)}\} .$$

The problem is, therefore, to find a sequence $C = \{C_t\}_{t=0}^{\infty}$ in $F(\eta_0 K_0)$ that maximizes W over $F(\eta_0 K_0)$.

3. Feasible paths

Our next task will be to show that the space of feasible paths $F(\eta_0 K_0)$ is naturally included in a Banach space, consisting of sequences that are summable in absolute value with respect to a *finite* measure on the space of integers. First, we define our Banach space H_λ and then we show in Lemmas 1 and 2 that all feasible paths are included in this space.

Let $\boldsymbol{Z}$ denote the set of integers and $\boldsymbol{R}$ the real line. Define the space of sequences $H_\lambda \equiv \{f : \boldsymbol{Z} \to \boldsymbol{R} : f(t) = f_t,\ \Sigma f_t \lambda^{-(t^2+t/2)} < \infty, \text{ for } \lambda > \eta_0 g(\gamma) > 1\}$. H_λ is a Banach space with the norm

$$\|f\| = \sum |f_t| \lambda^{-(t^2+t/2)} .$$

H_λ is a *weighted* L_1 space with the bounded measure $\lambda(t) = \lambda^{-(t^2+t/2)}$ for $t \in \mathbf{Z}$.

Lemma 1. For all sequences $\{K_t\}_{t=0}^{\infty}$ *and* $\{\eta_t\}_{t=0}^{\infty}$ *that satisfy* (2) *and* (3):

$$\begin{aligned} &\text{(i)} \quad K_t \leq \eta_0^t K_0 g(\gamma)^{t(t-1)/2} , \\ &\text{(ii)} \quad \eta_t \leq \eta_0 g(\gamma)^t , \\ &\text{(iii)} \quad \eta_t K_t \leq \eta_0^{t+1} K_0 g(\gamma)^{t(t+1)/2} , \end{aligned} \tag{6}$$

for $t = 0, 1, \ldots, \infty$. *In particular, all feasible consumption paths are in* H_λ, *i.e.* $F(\eta_0 K_0) \subset H_\lambda$.

Proof. Consider the path $\{\bar{\eta}_t, \bar{K}_t\}_{t=0}^{\infty}$ defined as follows:

$$\bar{\eta}_0 = \eta_0 , \qquad \bar{K}_0 = K_0 , \qquad \bar{\eta}_{t+1} = \bar{\eta}_t g(\gamma) \quad \text{and} \quad \bar{K}_{t+1} = \bar{\eta}_t \bar{K}_t$$

for all t. Note that this path is not feasible by (2). Furthermore,

$$\begin{aligned} \bar{K}_1 &= \eta_0 K_0 , \\ \bar{K}_2 &= \eta_1 K_1 = \eta_0^2 K_0 g(\gamma) , \\ \bar{K}_3 &= \eta_2 K_2 = \eta_0^3 K_0 g(\gamma)^3 , \\ \bar{K}_4 &= \eta_3 K_3 = \eta_0^4 K_0 g(\gamma)^6 . \end{aligned}$$

In fact, induction yields

$$\bar{K}_t = \eta_0^t K_0 g(\gamma)^{t(t-1)/2} ,$$

which proves (i). Part (ii) is obvious. From (i) and (ii) we obtain

$$\begin{aligned} \bar{\eta}_1 \bar{K}_1 &= \eta_0^2 K_0 g(\gamma) , \\ \bar{\eta}_2 \bar{K}_2 &= \eta_0^3 K_0 g(\gamma)^3 , \\ \bar{\eta}_3 \bar{K}_3 &= \eta_0^4 K_0 g(\gamma)^6 , \\ \bar{\eta}_4 \bar{K}_4 &= \eta_0^5 K_0 g(\gamma)^{10} . \end{aligned}$$

Again, induction yields

$$\bar{\eta}_t \bar{K}_t = \eta_0^{t+1} K_0 g(\gamma)^{t(t+1)/2} ,$$

which proves (iii), since $\bar{\eta}_t \bar{K}_t$ exceeds any path satisfying (2) and (3). □

Lemma 2. The set $F = F(\eta_0 K_0)$ *is a compact subset of* H_λ.

Proof. Let $B = \{B_t\}_{t=0}^{\infty}$ denote the sequence $\{\bar{\eta}_t \bar{K}_t\} = \{\eta_0^{t+1} K_0 g(\gamma)^{t(t+1)/2}\}$ from Lemma 1. Since $\forall C \in F(\eta_0 K_0)$, $C_t \leq \eta_t K_t$, it follows that $C_t \leq B_t$, $\forall t$. Note that $B \in H_\lambda$. This is because for $t \geq 2$

$$\eta_0^{t+1} K_0 g(\gamma)^{t(t+1)/2} < [\eta_0 g(\gamma)]^{t(t+1)/2} \eta_0 K_0 < \lambda^{t(t+1)/2} \eta_0 K_0 ,$$

since $\lambda < \eta_0 g(\gamma)$ by the construction of H_λ. This implies that $\Sigma_{t=0}^{\infty} B_t \lambda^{-(t^2+t/2)} < \infty$.

The set $F = F(\eta_0 K_0) \subset H_\lambda$ is bounded above by the sequence $B \in H_\lambda$, and is also bounded below by the zero sequence because consumption is positive. Furthermore, since the function g in (3) is continuous, F is a closed subset of H_λ. Since F is a norm-bounded subset of H_λ, by Banach Alaoglu's theorem, F is weak* compact. By definition, this means that every sequence $\{f^n\}$ in F has a subsequence that converges pointwise $f^n \to_w f^* \in F$. Since f^* is also bounded by B and $\Sigma_{t=0}^{\infty} B_t \lambda^{-(t^2+t/2)} < \infty$, by Lebesgue's bounded convergence theorem, the convergence is also in the norm of H_λ, i.e. $f^n \to_{\|\cdot\|} f^*$. It follows that $F(\eta_0 K_0)$ is compact in the norm of H_λ. □

4. The existence of an optimal growth path

Our next task is to prove that, although the welfare function W is not continuous on all of H_λ, and indeed it is not defined on any sequence that contains zero consumption, if we restrict the set of feasible paths to those that attain at least a given utility level, then the set of feasible paths that yield at least this level is compact and the welfare function is well defined and continuous on this set. Using this fact we prove the existence of an optimal growth path.

Lemma 3. There exists a consumption sequence $\{\tilde{C}\} \in F(\eta_0 K_0)$ *with* $W\{\tilde{C}\} = \Sigma_0^{\infty} \beta^{-t} \ln \tilde{C}_t > -\infty$.

Proof. Consider the feasible policy $\tilde{C} = \{\tilde{C}_t\} = (1/3)\tilde{\eta}_t \tilde{K}_t$, $\tilde{Z}_t = (1/3)\tilde{\eta}_t \tilde{K}_t$, and $\tilde{K}_{t+1} = (1/3)\tilde{\eta}_t \tilde{K}_t$ for all t with $\tilde{\eta}_0 = \eta_0$, $\tilde{K}_0 = K_0$. As in Lemma 1, straightforward calculation yields

$$\tilde{\eta}_1 \tilde{K}_1 = (1/3)\eta_0^2 K_0 g(\gamma/3) ,$$

$$\tilde{\eta}_2 \tilde{K}_2 = (1/3)^2 \eta_0^3 K_0 g(\gamma/3)^3 ,$$

$$\tilde{\eta}_3 \tilde{K}_3 = (1/3)^3 \eta_0^4 K_0 g(\gamma/3^6) .$$

By induction

$$\tilde{\eta}_t \tilde{K}_t = (1/3)^t \eta_0^{t+1} K_0 g(\gamma/3)^{t(t+1)/2} .$$

The utility from the consumption sequence $\tilde{C}$ is

$$W(\tilde{C}) = \frac{1}{(1-\beta^{-1})^2}\ln(1/3) + \frac{\beta}{(1-\beta^{-1})^2}\ln\eta_0 + \frac{1}{1-\beta^{-1}}\ln K_0 + \frac{\beta}{(1-\beta^{-1})^3}\ln g(\gamma/3) > -\infty\,. \quad \square$$

Lemma 4. Let $G_\epsilon = \{f \in F = F(\eta_0 K_0) : W(f) \geq W(\tilde{C}) - \epsilon$, *with* $\tilde{C}$ *as in Lemma 3. The function* $W(C) = \Sigma_0^\infty \beta^{-t}\ln C_t$ *is well defined and norm-continuous on* $G_\epsilon \subset H_\lambda$, $\forall \epsilon \geq 0$.

Proof. First, we show that W is well defined on G_ϵ. By Lemma 3, $G_\epsilon \neq \emptyset$. Let $x \in G_\epsilon$. Then, by the definition of G_ϵ, we know that $W(x) > -\infty$. We now show that $W(x) < \infty$. Since $x \in H_\lambda$, by definition $\Sigma_{t=0}^\infty x_t \lambda^{-(t^2+t/2)} < \infty$, which implies in particular that $\lim_{t\to\infty} x_t \lambda^{-(t^2+t/2)} = 0$. Therefore, for t large enough, $x_t < \lambda^{t^2+t/2}$ so that $\ln x_t < (t^2 + t/2)\ln\lambda$. Now $\Sigma_{t=0}^\infty \beta^{-t}(t^2 + t/2)\ln\lambda < \infty$ (Knopp, 1956). Therefore, W is well defined on G_ϵ. Next, we prove continuity of W on G_ϵ. Assume that $f^n \to f$ in the $\|\cdot\|$ norm. Then by definition $\lim_{n\to\infty}\Sigma_{t=0}^\infty (f_t^n - f_t)\lambda^{-(t^2+t/2)} = 0$. Since f^n and f are in G_ϵ, this implies that for $\bar{t}$ large enough the 'cut-off' sequences $g_t^n = f_t^n$, defined as $g_t = f_t$ for $t \leq \bar{t}$ and 0 otherwise, are also in the set G_ϵ. Therefore, $\Sigma_{t=0}^\infty \beta^{-t}|\ln g_t^n - \ln g_t| = \Sigma_{t=0}^{\bar{t}} \beta^{-t}|\ln f_t^n - \ln f_t| \to 0$ when $n \to \infty$. Furthermore, since both f^n and f are in H_λ^+, for t large enough, $f_t^n < \lambda^{(t^2+t/2)}$ and $f_t < \lambda^{(t^2+t/2)}$ so that $|\ln(f_t^n)| < (t^2 + t/2)\ln\lambda$ and $|\ln(f_t)| < (t^2 + t/2)\ln\lambda$. Therefore, $|\ln f_t^n - \ln f_t| < (2t^2 + t)\ln\lambda$ and thus $\Sigma_{t>\bar{t}}\beta^{-t}|\ln f_t^n - \ln f_t| < \Sigma_{t>\bar{t}}\beta^{-t}(2t^2+t)\ln\lambda$, which converges to 0 as $\bar{t}\to\infty$. We have, therefore, proven that $W(f^n) \to W(f)$, and therefore the function W is continuous on G_ϵ. $\square$

Theorem 1. There exists a feasible consumption path $\{C^*\}$ *that maximizes* $W(\cdot)$ *subject to* (2)–(5).

Proof. Let G_0 denote the set G_ϵ for $\epsilon = 0$. Note that $G_0 \subset G_\epsilon$. By Lemma 4, the set G_0 is closed because it is defined by means of the function $W(\cdot)$, which is continuous on $G_\epsilon \supset G_0$, and is a subset of $F = F(\eta_0 K_0)$, which is closed. Since $G_0 \subset F$ and F is $\|\cdot\|$ compact by Lemma 2, then G_0 is also $\|\cdot\|$ compact. Finally, $W(\cdot)$ is continuous on G_0 with the $\|\cdot\|$ norm, so that a maximum $\{C^*\} \subset G_0$ exists. $\square$

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